Wild singularities of flat surfaces

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Abstract. We consider flat surfaces and the points of their metric completions, particularly the singularities to which the flat structure of the surface does not extend. The local behavior near a singular point x can be partially described by a topological space $\mathcal{L}(x)$ which captures all the ways that x can be "approached linearly". The homeomorphism type of $\mathcal{L}(x)$ is an affine invariant. When x is not a cone point or an infinite-angle singularity, we say it is *wild*; in this case it is necessary to add further metric data to $\mathcal{L}(x)$ to get a quantitative description of the surface near x.

The study of flat surfaces, appearing under different guises (quadratic differentials, abelian differentials, translation surfaces, measured foliations, F-structures, and so on), reaches back at least to the 1970–80s, when seminal work of Thurston, Masur, Veech, and others uncovered fundamental connections among surface automorphisms, flat surface geometry, and billiard dynamics. However, their origins go back much further to the 1930–40s, with Nielsen's classification of torus automorphisms and Fox–Kerschner's association of a Riemann surface to billiards in a polygon [Fox36], sometimes called the Katok–Zemlyakov unfolding construction. Throughout much of the history of flat surfaces, the focus has been on *compact* flat surfaces, having so-called "cone-type" singularities, with non-compact surfaces appearing only sporadically. In this way, researchers could bring to bear the considerable power of finite-dimensionality in Teichmüller theory and in algebraic constructions such as homology groups.

In recent years, increasing attention has been paid to the study of non-compact flat surfaces, or more precisely surfaces of infinite type. Several treatments deal with classes of examples such as covers of compact surfaces [HWS] or surfaces arising from certain dynamical systems (wind-tree models [HLT], irrational billiards [Val], exchanges of infinitely many intervals [Hoo10], etc.). In a similar vein, de Carvalho–Hall have initiated a study of dynamical systems on genus-zero surfaces with infinitely many singularities [dCH11]. These studies have necessitated the adaptation of tools from the theory of compact flat surfaces, but have so far remained fuzzy on the local, intrinsic behavior of a surface near its singular points. The simple description via cone points becomes inadequate when the metric structure imposed on a surface can allow for essentially arbitrary topological complication within a bounded region. In our opinion, this constitutes an important lack and an obstacle to properly understanding basic notions such as straight-line flow and deformations of flat surfaces.

Here we present a method for studying the local behavior of singularities of topologically infinite flat surfaces. For the most part, we restrict our attention to *isolated* singularities in the metric completion of a flat surface. These singularities do not, in general, have an analytic description parallel to the description of cone points as zeroes of holomorphic differentials. Nor are they determined (up to local isometry) by discrete sets of data.

Our primary invariant is a topological space $\mathcal{L}(x)$ associated to each point x in the metric completion of a flat surface, which may be thought of as a set of directions arising from x; each element of $\mathcal{L}(x)$ represents a "linear approach" to x. This space is invariant under affine deformations of the surface. We then add metric data to $\mathcal{L}(x)$ that quantify how the linear approaches are distributed. Together, these provide a complete description of the surface near x.

From our perspective, this paper provides a unifying vision of the disparate singular behaviors that had previously been only superficially observed. In §1 we recall some motivating examples and define the invariant $\mathcal{L}(x)$ along with its global version $\mathcal{L}(X)$, where X is a flat surface and x is in the metric completion of X. In §2 we examine the topological structure of $\mathcal{L}(X)$ and $\mathcal{L}(x)$ in more detail. We prove in particular that the space $\mathcal{L}(X)$ is an extension of the unit tangent bundle of the flat surface X to its metric completion. In §3 we study the effect of affine maps on $\mathcal{L}(X)$ and $\mathcal{L}(x)$. In §4 we provide a set of necessary and sufficient conditions for two points to have isometric neighborhoods. Finally, in §5 we briefly compare our invariants with similar constructions that have been previously described. Our constructions are quite general and would apply in many contexts outside of flat surfaces, while they also retain extra available information due to properties of flat surfaces that set them apart from general metric spaces.

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1. Basic definitions and examples.

In this section we introduce the basic definitions and examples. The main objects we will be working with are translation surfaces arising from holomorphic 1-forms on a fixed Riemann surface.

Definition 1.1. A *flat surface* is a pair (X, ω) formed by a Riemann surface X and a non identically zero holomorphic 1-form ω on X. Where clarity permits, we abbreviate (X, ω) by X. We denote by $Z(\omega) \subset X$ the set of zeroes of ω.

Sometimes we will also use the terminology *translation surface* to refer to a flat surface. Local integration of the form ω endows $X' := X \setminus Z(\omega)$ with an atlas whose transition functions are translations of $\mathbb{C}$. The pullback of the standard translation invariant flat metric on the complex plane defines a flat metric d_X on $X \setminus Z(\omega)$. We will denote by $\widehat{X}$ the metric completion of X' with respect to d_X. In this article we will work with flat surfaces satisfying the following:

Main hypothesis. *The set* $\mathrm{Sing}(X) := \widehat{X} \setminus X'$ *is a discrete subset of* $\widehat{X}$.

Remark that $\widehat{X}$, and hence $\mathrm{Sing}(X)$, depends on our choice of the 1-form ω on X. Points in $\mathrm{Sing}(X)$ fall in one of the following cases.

(1) Flat points. These are points $p \in \widehat{X} \setminus X$ for which the flat metric of X extends to a flat metric on $X \cup \{p\}$.
(2) Finite angle singularities. These are points $p \in \widehat{X}$ for which the Riemann surface structure of X extends to $X \cup \{p\}$. In a neighborhood of p the form ω is given by $z^k\, \mathrm{d}z$ for some $k \in \mathbb{N}$.
(3) Infinite angle singularities. For each of these singularities $p \in \widehat{X}$, there exists a punctured neighborhood $0 < d_X(w,p) < \varepsilon$ which is isometric to an infinite cyclic covering of the punctured disc $(0 < |z| < \varepsilon, \mathrm{d}z)$. Such punctured neighborhoods can be pictured as an infinite double helicoid whose axis has been collapsed to a point. Infinite angle singularities naturally appear in flat surfaces associated to irrational polygonal billiards.
(4) The rest. We call such points p *wild* singularities of the flat surface. These points and their neighborhoods $0 < d_X(w,p) < \varepsilon$ will constitute the main research point of this article.

Convention. Henceforth we will work only with flat surfaces (X,ω) such that the set of flat points in $\mathrm{Sing}(X)$ is empty.

Definition 1.2 (Saddle connection)**.** A *critical trajectory* of a flat surface (X,ω) is an open geodesic in the flat metric d_X whose image under the natural embedding $X \hookrightarrow \widehat{X}$ issues from a point in $\mathrm{Sing}(X)$, contains no other point of $\mathrm{Sing}(X)$ in its interior and is not property contained in some other geodesic segment. A *saddle connection* is a finite length critical trajectory.

Definition 1.3 (Veech group)**.** Let $\mathrm{Aff}_+(X,\omega)$ be the group of affine orientation preserving homeomorphisms of (X,ω). Consider the map that associates to each $\varphi \in \mathrm{Aff}_+(X,\omega)$ its Jacobian derivative $D\varphi \in \mathrm{GL}_+(2,\mathbb{R})$. We call the image of this map the *Veech group* of (X,ω) and denote it by $\Gamma(X)$.

In the following paragraphs we introduce the topological spaces $\mathcal{L}(X)$ and $\mathcal{L}(x)$. They will play the role of unit tangent bundle and unit tangent space on $\widehat{X}$ respectively.

Definition 1.4 (Linear approach)**.** Given $\varepsilon > 0$, let $\mathcal{L}^\varepsilon(X)$ be the space

$$\mathcal{L}^\varepsilon(X) := \{\text{unit speed geodesic trajectories } \gamma : (0,\varepsilon) \to X'\}.$$

Each $\mathcal{L}^\varepsilon(X)$ carries the uniform topology, defined by the uniform metric:

$$d_\varepsilon(\gamma_1,\gamma_2) = \sup_{0<t<\varepsilon} d_X\big(\gamma_1(t),\gamma_2(t)\big)$$

Two elements $\gamma_1 \in \mathcal{L}^\varepsilon(X)$ and $\gamma_2 \in \mathcal{L}^{\varepsilon'}(X)$ are said to be equivalent if and only if $\gamma_1(t) = \gamma_2(t)$ for all $t \in (0, \min\{\varepsilon,\varepsilon'\})$. We denote by $\sim$ this equivalence relation and define:

$$\mathcal{L}(X) := \bigsqcup_{\varepsilon>0} \mathcal{L}^\varepsilon(X)/\sim \tag{1.1}$$

The equivalence class of γ will be denoted by $[\gamma]$. We call each element $[\gamma]$ of $\mathcal{L}(X)$ a *linear approach* to the point $\lim_{t\to 0}\gamma(t) \in \widehat{X}$.

Topology for $\mathcal{L}(X)$. For each $\varepsilon' \leq \varepsilon$ the restriction of each linear approach in $\mathcal{L}^\varepsilon(X)$ to the interval $(0, \varepsilon')$ defines a continuous injection:

$$\rho_\varepsilon^{\varepsilon'} : \mathcal{L}^\varepsilon(X) \to \mathcal{L}^{\varepsilon'}(X)$$

Define $\varepsilon \trianglelefteq \varepsilon'$ if and only if $\varepsilon' \leq \varepsilon$, where $\leq$ is the standard order in $\mathbb{R}$. Then $\langle \mathcal{L}^\varepsilon(X), \rho_\varepsilon^{\varepsilon'} \rangle$ is a direct system of topological spaces over $(\mathbb{R}^+, \trianglelefteq)$. Since for every $\varepsilon \trianglelefteq \varepsilon' \trianglelefteq 0$ the projection map $\gamma \mapsto [\gamma]$ from $\mathcal{L}^\varepsilon(X)$ to $\mathcal{L}(X)$ is injective and commutes with $\rho_\varepsilon^{\varepsilon'}$, we have the equality of sets

$$\mathcal{L}(X) = \varinjlim \mathcal{L}^\varepsilon(X) \tag{1.2}$$

Henceforth we endow $\mathcal{L}(X)$ with the *direct limit topology* (sometimes called the *final topology*), which is the finest topology such that the inclusions $\mathcal{L}^\varepsilon(X) \to \mathcal{L}(X)$ are all continuous. We denote by $\rho_\varepsilon : \mathcal{L}^\varepsilon(X) \to \mathcal{L}(X)$ the natural projection $\gamma \mapsto [\gamma]$. Unless otherwise stated, we also identify $\mathcal{L}^\varepsilon(X)$ with its image in $\mathcal{L}(X)$.

Definition 1.5. Let $x \in \widehat{X}$. We define $\mathcal{L}(x)$ to be the set of all linear approaches $[\gamma] \in \mathcal{L}(X)$ such that $\lim_{t\to 0} \gamma(t) = x$, endowed with the subspace topology.

The space $\mathcal{L}(x)$ naturally decomposes into rotational components, which we define in the following paragraphs.

Definition 1.6 (Angular sector)**.** We call an *angular sector* a triple of the form (I, c, i_c), where $I \subseteq \mathbb{R}$ is a non-empty *interval*, $c \in \mathbb{R}$ is a constant and i_c is a isometry into X' of the open set

$$U = U(I, c) := \{(x, y) \mid x < c,\, y \in I\}, \tag{1.3}$$

endowed with the translation structure defined by the holomorphic 1-form $e^z\,\mathrm{d}z$ (where $z = x + iy$).

Observe that for every fixed angular sector (I, c, i_c) the limit $\lim_{x\to-\infty} i_c(x, y)$ exists in $\widehat{X}$ and is independent from the y-coordinate in $U(I, c)$ into X.

Convention: All sets U are contained in the same copy of $\mathbb{R}^2$ on which we have previously fixed our favorite orientation. The interval I in the preceding definition can be just a point, unbounded and are not necessarily closed or open.

Definition 1.7 (Rotational component)**.** Let $[\gamma_1]$ and $[\gamma_2]$ be two linear approaches in $\mathcal{L}(x)$. We say that $[\gamma_1]$ and $[\gamma_2]$ are equivalent if and only if there exist representatives $\gamma_i : (0, \varepsilon_i) \to X$, $i = 1, 2$ and an angular sector (I, c, i_c) such that $(i_c^{-1} \circ \gamma_i)(0, \varepsilon_i)$ is equal to an infinite segment of real line $(x < c, y_i)$, for some fixed $y_i \in I$, $i = 1, 2$. We denote by $\overline{[\gamma]}$ the equivalence class defined by $[\gamma] \in \mathcal{L}(x)$, and we call this class *the rotational component of* $\mathcal{L}(x)$ *containing* $[\gamma]$.

Lemma 1.8. *Every rotational component $\overline{[\gamma]}$ containing more than one element admits a connected real 1-manifold translation structure, possibly with non-empty boundary.*

Proof. For every angular sector (I, c, i_c) making two linear approaches in $\overline{[\gamma]}$ equivalent, we define

$$V = V(I, c, i_c) = \{i_c(x, y) \mid x < c\,\}_{y \in I} \quad \subseteq \quad \overline{[\gamma]}.$$

Call $\mathcal{V}$ the collection of all the $V(I, c, i_c)$ obtained by considering angular sectors (I, c, i_c) as before. This collection is the basis for a topology on $\overline{[\gamma]}$. With respect

to this topology the class $\overline{[\gamma]}$ is Hausdorff, second countable and connected. Define $\varphi_V : V \to I$ by $\varphi_V[i_c(x < c, y)] = y$. This is a local homeomorphism. Given the convention made after definition 1.6, the set $\{(V, \varphi_V)\}_{V\in\mathcal{V}}$ defines an atlas on $\overline{[\gamma]}$ whose transition functions are translations in $\mathbb{R}$. Remark that charts for boundary points are defined by left or right closed intervals. □

Remark 1.9. From now on $\overline{[\gamma]}$ will denote the rotational component defined by the linear approach $[\gamma]$ and endowed with the translation structure given by the preceding proposition. We can lift the standard translation invariant metric of $\mathbb{R}$ to each $\overline{[\gamma]}$. We have the following situations:

(1) If $\mathcal{L}(x)$ contains a compact rotational component $\overline{[\gamma]}$, there are two possibilities:
  (1.a) The rotational component is an interval $[a, b]$, perhaps with $a = b$. In this case the rotational component is a proper subset of $\mathcal{L}(x)$.
  (1.b) The rotational component is homeomorphic to S^1. In this case $\mathcal{L}(x)$ and $\overline{[\gamma]}$ are homeomorphic as topological spaces.

(2) If $\mathcal{L}(x)$ contains a non compact rotational component $\overline{[\gamma]}$. The following situations can occur:
  (2.a) The total length of $\overline{[\gamma]}$ is finite. In this case the rotational component is isometric to a bounded interval and is a proper subset of $\mathcal{L}(x)$.
  (2.b) The total length of $\overline{[\gamma]}$ is infinite, but the class is isometric to an unbounded proper interval of $\mathbb{R}$. In this case we say that the class $\overline{[\gamma]}$ is a *spire*. A spire may coincide with or be a proper subset of $\mathcal{L}(x)$.
  (2.c) The total length of $\overline{[\gamma]}$ is infinite, and the class is isometric to $\mathbb{R}$. In this case we say that the class $\overline{[\gamma]}$ is a *double spire*. This case contains all infinite angle singularities, for which necessarily $\mathcal{L}(x) = \overline{[\gamma]}$. Nevertheless, there are examples for which $\mathcal{L}(x)$ is a double spire *but x is not* an infinite angle singularity.

Remark 1.10. Let $x \in \widehat{X}$ be a flat point, a finite angle singularity of angle $2\pi k$, $k > 1$, or an infinite angle singularity. Then $\mathcal{L}(x)$ is isometric to $\mathbb{R}/2\pi\mathbb{Z}$, $\mathbb{R}/2\pi k\mathbb{Z}$ or $\mathbb{R}$ respectively.

1.1. **Examples.** In the rest of this section we present some examples of translation surfaces having wild singularities.

Example 1.11 ([Cha04, CGL06])**.** This example appears naturally when studying the self mappings of the unit square known as the *horseshoe* and *baker's* map. Start with a unit square S, let $\alpha = 1/2$ and partition its top edge into segments of lengths α^k, $1 \le k < \infty$, in decreasing order from left to right. Do the same for the bottom edge, but in reverse order. Remove extremities of all segments involved and identify (open) segments of the same length via translation. Now partition the left edge from top to bottom in the same way, and the right edge from bottom to top, remove extremities of all segments involved and identify those of the same length via translation. (See Figure 1.) The result is an open flat Riemann surface X_α of infinite genus with one end (*a.k.a.* a Loch Ness monster after Ghys [Ghy95]). The metric completion $\overline{X_\alpha}$ is obtained by adding the extremities of each A_i and B_i, $i \in \mathbb{N}$. In $\overline{X_\alpha}$ all this added points are at distance zero from each other, hence $\overline{X_\alpha} \setminus X_\alpha = x$. The horizontal and vertical flow on X_α define two families of saddle connections whose length is not bounded away from zero.

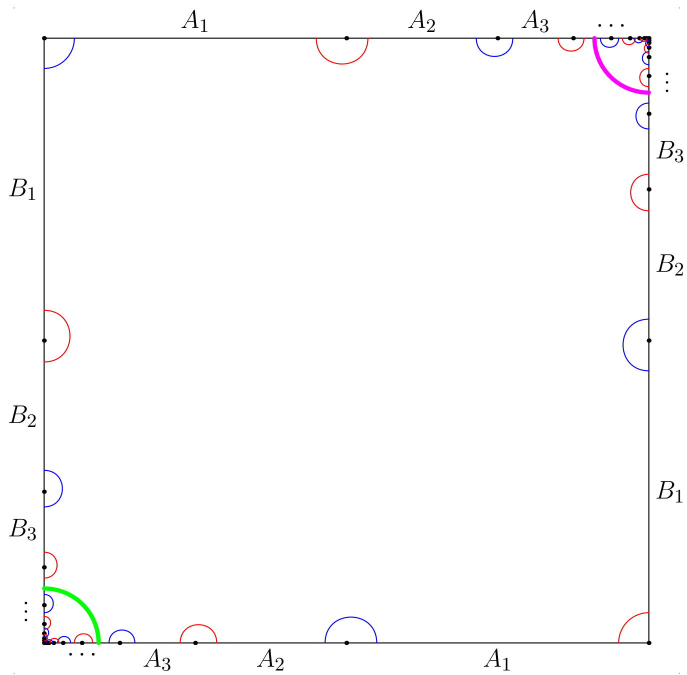

FIGURE 1. Double spires and finite length rotational components in $X_{1/2}$.

Therefore x is a wild singularity. Chamanara observes that "[g]eometrically, the surface spirals infinitely many times around this point." In fact, $\mathcal{L}(x)$ decomposes into an infinite number of rotational components. Indeed, if the intersection of the diagonals in the unit square is the origin and points in $A_1 \cap B_1$ are $(-\frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, -\frac{1}{2})$, then $\gamma_1(t) := (1-t)(-\frac{1}{2}, \frac{1}{2})$ and $\gamma_2(t) := -\gamma_1(t)$, define two rotational components which are double spires. On the other hand, $\eta_1(t) = (1-t)(\frac{1}{2}, \frac{1}{2})$ and $\eta_2(t) = -\eta_1(t)$, define two rotational components whose length is $\pi/4$. Remark that the obstructions for these rotational components to become spires are precisely the horizontal and vertical saddle connections defined by A_i and B_i, $i \in \mathbb{N}$. Using Chamanara's results [Cha04, Theorem B] about the Veech group of X_α one can prove that $\mathcal{L}(x)$ decomposes into countably many rotational components. The preceding argumentation remains valid if we change $\alpha = 1/2$ for $\alpha = 1/n$ with $n \in \mathbb{N}$.

Example 1.12 (The geometric series construction)**.** We introduce a local construction that will provide an archetype for half spires. Let $0 < \alpha < 1$, $I_0 = [0, \alpha]$ and, for each $n \geq 1$ define $I_n = [\sum_{m=1}^{n} \alpha^m, \sum_{m=1}^{n+1} \alpha^m]$. Let $J_0 = [1-\alpha, 1]$ and $J_n = [1 - \sum_{m=1}^{n+1} \alpha^m, 1 - \sum_{m=1}^{n} \alpha^m]$, $n \geq 1$. Now slit the xy-plane along $[0,1]$. Points in the boundary of the slitted plane are thought as two different copies of $[0,1]$, each of which we identify with the families of segments $\cup_{n\geq 0} I_n$ and $\cup_{n \geq 0} J_n$ respectively. Using translations, we identify for each $n \geq 0$ the segment I_n with J_n and remove the extremities of the segments involved at each step. The result

is a flat surface Y_α such that $\mathrm{Sing}(\widehat{Y_\alpha}) = x$. The space $\mathcal{L}(x)$ is formed by two rotational components whose representatives are the linear approaches $\gamma_1(t) = (0,t)$ and $\gamma_2(t) = (1,t)$, $t \in (0,1)$. Each the rotational component is isometric to $(0,\infty)$.

Example 1.13 (Double parabola)**.** In this example we construct rotational components consisting of only one point. Let $\pm I_n$ be a family of segments in the xy-plane whose endpoints are $(\pm 2^n, 2^{2n})$ and $(\pm 2^{n+1}, 2^{2(n+1)})$, $n \in \mathbb{Z}$. Let $\pm J_n$ be the family of segments whose endpoints are $(\pm 2^n, -2^{2n})$ and $(\pm 2^{n+1}, -2^{2(n+1)})$, $n \in \mathbb{Z}$. Let P_- be closure of the connected component of $\mathbb{R}^2 \setminus \{\pm I_n\}_{n\in\mathbb{Z}} \cup (0,0)$ containing the negative x-axis. Analogously, let P_+ be the closure of the connected component of $\mathbb{R}^2 \setminus \{\pm J_n\}_{n\in\mathbb{Z}} \cup (0,0)$ containing the positive x-axis. By construction $\partial P_- = \{-I_n\}_{n\in\mathbb{Z}} \cup (0,0) \cup \{-J_n\}_{n\in\mathbb{Z}}$ and $\partial P_+ = \{I_n\}_{n\in\mathbb{Z}} \cup (0,0) \cup \{J_n\}_{n\in\mathbb{Z}}$. Remove all vertices (and the origin) from P_- and P_+ and identify this two disjoint domains along parallel sides of the same length using translations. The result of this construction is a flat surface X for which $\mathrm{Sing}(\widehat{X})$ is only one wild singularity x. The rotational components defined by $\pm\gamma(t) = (\pm t, 0)$ consist of only one point. It is easy to check that in this case $\mathcal{L}(x)$ contains also two double spires.

Other examples of translation surfaces with wild singularities include Hooper's generalization of Thurston's construction to infinite bipartite graphs [Hoo10] or the geometric limit of the Arnoux–Yoccoz family studied by the first author [Bow11].

2. Topology of $\mathcal{L}(X)$ and $\mathcal{L}(x)$

2.1. **Universal property of the direct limit.** In the previous section, we defined $\mathcal{L}(X)$ as the direct limit of the topological spaces $\mathcal{L}^\varepsilon(X)$ and thereby endowed $\mathcal{L}(x)$ with the direct limit topology. In this section and the next we will explore some of the consequences of this topology. First, we state the corresponding universal property of $\mathcal{L}(X)$. Recall that for $\varepsilon' < \varepsilon$, $\rho_\varepsilon^{\varepsilon'}$ is the restriction map $\mathcal{L}^\varepsilon(X) \to \mathcal{L}^{\varepsilon'}(X)$, and for all $\varepsilon > 0$, ρ_ε is the map $\gamma \mapsto [\gamma]$.

Universal Property of $\mathcal{L}(X)$. *Let X be a translation surface. Given any topological space $\mathfrak{T}$ and any collection of continuous maps $\{f_\varepsilon : \mathcal{L}^\varepsilon(X) \to \mathfrak{T}\}_{\varepsilon>0}$ such that $f_{\varepsilon'} \circ \rho_\varepsilon^{\varepsilon'} = f_\varepsilon$ whenever $\varepsilon' < \varepsilon$, there is a unique continuous map $f : \mathcal{L}(X) \to \mathfrak{T}$ such that $f \circ \rho_\varepsilon = f_\varepsilon$ for all ε.*

2.2. **Continuity of the basepoint and direction maps.** Two functions we would like to define on $\mathcal{L}(X)$ are the *basepoint* and *direction* maps, respectively denoted bp and dir, and given on $\mathcal{L}^\varepsilon(X)$ by

$$\begin{aligned} &\mathrm{bp} : \gamma \mapsto \lim_{t\to 0} \gamma(t) \in \widehat{X} \\ &\mathrm{dir} : \gamma \mapsto \dot\gamma(t) \in S^1 \qquad\qquad \text{for any } t \in (0,\varepsilon). \end{aligned}$$

The basepoint map satisfies $\mathrm{bp} \circ \rho_\varepsilon^{\varepsilon'} = \mathrm{bp}$ because it does not depend on the length of the domain of γ, only on its values near zero. The direction map is well-defined because the translation structure of X yields a trivialization of the unit tangent bundle $T_1(X) = X \times S^1$. Since each geodesic is contained in a fiber of the projection $X \times S^1 \to S^1$ the direction map satisfies $\mathrm{dir} \circ \rho_\varepsilon^{\varepsilon'} = \mathrm{dir}$.

Proposition 2.1. *For any $\varepsilon > 0$, the functions* $\mathrm{bp} : \mathcal{L}^\varepsilon(X) \to \widehat{X}$ *and* $\mathrm{dir} : \mathcal{L}^\varepsilon(X) \to S^1$ *are continuous.*

Proof. Let $x \in \widehat{X}$ and $r > 0$. Suppose $d_X(x, \mathrm{bp}(\gamma)) < r$, and set $r' = r - d_X(x, \mathrm{bp}(\gamma))$. Then for any $\eta \in \mathcal{L}^\varepsilon(X)$ such that $d_\varepsilon(\gamma, \eta) < r'$,

$$d_X(x, \mathrm{bp}(\eta)) \leq d_X(x, \mathrm{bp}(\gamma)) + d_\varepsilon(\gamma, \eta) < d_X(x, \mathrm{bp}(\gamma)) + r' = r$$

Thus the preimage of the open ball $B(x, r)$ is open in $\mathcal{L}^\varepsilon(X)$. Therefore the basepoint map is continuous. To show that the direction map is continuous, fix $\gamma \in \mathcal{L}^\varepsilon(X)$ and $\theta \in (0, \pi/2)$. Choose a and b such that $0 < a < b < \varepsilon$. The segment $\gamma([a, b]) \subset X'$ is compact. Hence we can find $\delta > 0$ such that $U := \bigcup_{t \in [a,b]} B(\gamma(t), \delta)$ is isometric to a Euclidean rectangle capped by two half discs of arbitrary small area. (One can think of U as the barrel of a gun, through which we want to aim trajectories sufficiently close to γ.) We develop U in the plane, where elementary geometry shows that δ can be chosen so that for every η with $d_\varepsilon(\eta, \gamma) < \delta$ the direction $\mathrm{dir}(\eta)$ lies in a neighborhood centered at $\mathrm{dir}(\gamma)$ of radius θ. □

Now the universal property of $\mathcal{L}(X)$ implies the following:

Corollary 2.2. *The functions* $\mathrm{bp} : \mathcal{L}(X) \to \widehat{X}$ *and* $\mathrm{dir} : \mathcal{L}(X) \to S^1$ *are continuous.*

2.3. **A generating set for the topology on $\mathcal{L}(X)$.** The definition of $\mathcal{L}(X)$ as a direct limit makes its topology somewhat obscure. In this section we introduce a generating set for this topology that will be useful, as we will see later, to prove topological statements about $\mathcal{L}(X)$.

For any $x \in X$ and $r > 0$, let $B(x, r)$ denote the open d_X-ball in X centered at x and having radius r. Then, for any $t > 0$, we set

$$\tilde{B}(x, r)^t = \big\{[\gamma] \in \mathcal{L}(X) \mid \gamma(t) \in B(x, r)\big\}.$$

Implicit in this definition is the assumption that, in order for $[\gamma] \in \tilde{B}(x, r)^t$, $[\gamma]$ must have a representative of length greater than t. The following technical claim is our main result for this section.

Proposition 2.3. *The collection of sets*

$$\mathcal{B} := \big\{\tilde{B}(x, r)^t \mid x \in X',\ r > 0,\ t > 0\big\} \tag{2.4}$$

generates the limit topology in $\mathcal{L}(X)$.

First, we describe the restriction of this topology to each of the spaces $\mathcal{L}^\varepsilon(X)$. Recall that we denote by d_ε the uniform metric on $\mathcal{L}^\varepsilon(X)$.

Lemma 2.4. *Let τ_ε denote the topology on $\mathcal{L}^\varepsilon(X)$ induced by d_ε. Then*

$$\mathcal{B}^\varepsilon := \big\{\tilde{B}(x, r)^t \mid x \in X',\ 0 < r,\ 0 < t < \varepsilon\big\}$$

generates τ_ε.

Proof. This is a straightforward variant of the well-known fact that the uniform topology on a collection of maps from one metric space to another coincides with the compact-open topology. □

Proof of Proposition 2.3. Clearly $\mathcal{B}$ defines a covering of $\mathcal{L}(X)$. We denote by τ' the topology generated by $\mathcal{B}$ and by τ the limit topology on $\mathcal{L}(X)$. The proof requires two steps.

Step 1: $\tau' \subseteq \tau$. That is, every element of $\mathcal{B}$ is open in the direct limit topology. This means we need to show that the intersection of each $\tilde{B}(x, r)^t \in \mathcal{B}$ with $\mathcal{L}^\varepsilon(X) \subset$

$\mathcal{L}(X)$ is open for every $\varepsilon > 0$. If $t < \varepsilon$, then by Lemma 2.4 we are done. If, on the other hand, $t \geq \varepsilon$, then given $[\gamma] \in \tilde{B}(x,r)^t$ we need to find an open set in $\tilde{B}(x,r)^t \cap \mathcal{L}^\varepsilon(X)$ containing $[\gamma]$. Choose a such that $0 < a < \varepsilon$ and take $\delta < r$ such that, as in the proof of Proposition 2.1, the δ-neighborhood U of $\gamma([a,t])$ is isometric to a Euclidean rectangle capped by half-discs. Choose times $t_1, \dots, t_n$ and radii $r_1, \dots, r_n$ so that each $B(\gamma(t_n), r_n)$ lies in U and $\{B(\gamma(t_n), r_n)\}$ covers $\gamma([a,\varepsilon])$. Then every trajectory in $\bigcap_{i=1}^n \tilde{B}(\gamma(t_n), r_n)^{t_n}$ remains in U from time a to ε. Let $\alpha \subset S^1$ be the arc around $\mathrm{dir}(\gamma)$ of length $\arctan(\delta/t)$. Then, by continuity of the direction map, $\mathrm{dir}^{-1}(\alpha)$ is open in $\mathcal{L}^\varepsilon(X)$. Hence

$$\left(\bigcap_{i=1}^n \tilde{B}(\gamma(t_n), r_n)^{t_n}\right) \cap \mathrm{dir}^{-1}(\alpha)$$

is an open set in $\tilde{B}(x,r)^t \cap \mathcal{L}^\varepsilon(X)$ containing $[\gamma]$.

Step 2: $\tau \subseteq \tau'$. Let $V \in \tau$. We need to show that each element of V is contained in a finite intersection $\bigcap U_i$ of sets U_i in $\mathcal{B}$, such that this intersection is itself contained in V. Choose $[\gamma] \in V$ having a representative $\gamma \in \mathcal{L}^\varepsilon(X)$. Identify $\rho_\varepsilon^{-1}(V)$ with $V_\varepsilon := V \cap \mathcal{L}^\varepsilon(X) \in \tau_\varepsilon$. Using the generating set for τ_ε provided by Lemma 2.4, we can find a finite collection $\{\tilde{B}_\varepsilon(x_i, r_i)^{t_i}\}_{i=1}^n$ such that the intersection I of its elements satisfies $[\gamma] \in I \subset V \cap \mathcal{L}^\varepsilon(X) \subset V$. Hence $\tau \subset \tau'$. $\square$

2.4. **Topological consequences.** In this section we explore some topological properties of $\mathcal{L}(X)$. Recall that we identify $\mathcal{L}^\varepsilon(X)$ with its image in $\mathcal{L}(X)$ by ρ_ε.

Corollary 2.5. *For all $t > 0$, $\bigcup_{\varepsilon > t} \mathcal{L}^\varepsilon(X)$ is open in $\mathcal{L}(X)$.*

Proof. If $[\gamma]$ has a representative γ of length $\varepsilon > t$ and t' satisfies $t < t' < \varepsilon$, then there exists some $r > 0$ such that $\tilde{B}(\gamma(t'), r)^{t'}$ lies in $\mathcal{L}^\varepsilon(X) \subset \mathcal{L}(X)$, contains $[\gamma]$ and consists of germs of trajectories $[\eta]$ having representatives of length greater than t and satisfying the condition $d_X(x, \eta(t)) < r$. $\square$

Definition 2.6 (Maximal length function)**.** Let ℓ be defined on $\mathcal{L}(X)$ by

$$\ell[\gamma] = \sup\left\{\varepsilon > 0 \mid \eta \in \mathcal{L}^\varepsilon(X),\ \rho_\varepsilon(\eta) = [\gamma]\right\}.$$

This is the *maximal length function*; it measures the longest a representative of the class $[\gamma]$ can be. It takes values in the positive extended reals.

Corollary 2.7. *The function $\ell : \mathcal{L}(X) \to (0, \infty]$ is lower semi-continuous.*

We recall that lower semi-continuity of a real-valued function φ on a topological space $\mathfrak{T}$ means either of the following (equivalent) properties holds:

- For all $t \in \mathbb{R}$, $\varphi^{-1}((t, \infty))$ is open in $\mathfrak{T}$.
- If $\{x_n\}_{n=1}^\infty$ is a sequence of points in $\mathfrak{T}$ such that $x_n \to x_\infty$, then $\varphi(x_\infty) \leq \liminf_{n\to\infty} \varphi(x_n)$.

Using the standard proof of the Extreme Value Theorem for continuous real-valued functions on compact spaces, one can see that a lower semi-continuous function achieves a minimum value on any compact subset of its domain.

Corollary 2.8. *Sequences in $\mathcal{L}(X)$ whose lengths tend to zero have no accumulation points in $\mathcal{L}(X)$, and thus they form closed subsets.*

Proof. Suppose $\{[\gamma_n]\}_{n=1}^{\infty}$ is a sequence of elements of $\mathcal{L}(X)$ such that $\ell[\gamma_n] \to 0$, and $[\gamma]$ is any element of $\mathcal{L}(X)$. Then $[\gamma]$ has a representative of length $\varepsilon > 0$, and for any $t \in (0, \varepsilon)$, there exists some N such that, for all $n \geq N$, $[\gamma_n]$ does not have a representative of length at least t, and so $[\gamma_n] \notin \tilde{B}(\gamma(t), r)^t$ for r small enough and $n \geq N$. $\square$

In particular, any sequence of linear approaches determined by saddle connections whose lengths tend to zero form a closed set in $\mathcal{L}(X)$.

Corollary 2.9. *$\mathcal{L}(X)$ is Hausdorff.*

Proof. Suppose $[\gamma_1] \neq [\gamma_2]$. Then there exist representatives γ_1 and γ_2 of the respective classes and some t in their common domain such that $\gamma_1(t) \neq \gamma_2(t)$. Set $r = \frac{1}{2} d_X(\gamma_1(t), \gamma_2(t))$. Then $\tilde{B}(\gamma_1(t), r)^t$ and $\tilde{B}(\gamma_2(t), r)^t$ are disjoint open sets containing $[\gamma_1]$ and $[\gamma_2]$, respectively. $\square$

Remark 2.10. The space $\mathcal{L}(X)$ is not in general metrizable. Indeed, it is not even regular. Recall that a topological space is *regular* if any point and any closed subset can be separated by disjoint open neighborhoods. For example, consider as X the geometric construction performed on $\mathbb{R}^2$ as in §1, Example 1.12. Any neighborhood of the horizontal trajectory emanating from the far right of the picture intersects any open set containing the sequence of saddle connections determined by the segments I_n. But, as observed following Corollary 2.8, these saddle connections form a closed set.

Corollary 2.11. *$\mathcal{L}(X)$ is second-countable.*

Proof. X itself is second-countable because it is a Riemann surface, and so it has a countable dense subset S. The sets $\tilde{B}(x, r)^t$, where $x \in S$, $r \in \mathbb{Q}$, and $t \in \mathbb{Q}$ (with $r > 0$, $t > 0$) generate the topology of $\mathcal{L}(X)$. $\square$

The following proposition shows that the space $\mathcal{L}(X)$ is an extension of the unit tangent bundle of the flat surface X.

Proposition 2.12. *There is a natural topological embedding of the unit tangent bundle of X' into $\mathcal{L}(X)$.*

Proof. We denote the unit tangent bundle of X' by $T^1(X')$. Given that X' is a translation surface, $T^1(X') = X' \times S^1$. For every $(x, \theta) \in T^1(X') \times S^1$ let $i(x, \theta) := [\gamma] \in \mathcal{L}(X)$ be such that $\mathrm{bp}(\gamma) = x$ and $\mathrm{dir}(\gamma) = \theta$. Injectivity for i follows from the fact that x is a flat point. We now show that i is a topological embedding.

Given that $X' \times S^1 \hookrightarrow \widehat{X} \times S^1$ is a topological embedding and $\mathrm{bp} \times \mathrm{dir} : \mathcal{L}(X) \to \widehat{X} \times S^1$ is continuous, the product topology of $X' \times S^1$ is contained in the subspace topology of $i(X' \times S^1)$. Hence it is sufficient to show that if $x \in X$, $t, r > 0$ and $[\gamma] \in \tilde{B}(x, r)^t \cap i(T^1(X'))$ are fixed, then there exists an open set $U \times V \subset X' \times S^1$ such that:

$$[\gamma] \in i(U \times V) \subset \tilde{B}(x, r)^t \cap i(T_1(X')).$$

Since $\lim_{t \to 0} \gamma(t) = x$ is a flat point, we can cover $\{x\} \cup \{\gamma(s) \mid s \in (0, t + \delta)\}$, for small $\delta > 0$, with finitely many d_x-balls of fixed radius $r' > 0$ whose union lies in X'. The existence of $U \times V$ follows from the fact that $r' > 0$ can be chosen arbitrary small. $\square$

3. Affine maps and rotational components

In the preceding sections we saw how the space $\mathcal{L}(X)$ generalizes the notion of unit tangent bundle. In this section we describe first how an affine map between two translation surfaces X and Y induces a continuous map from $\mathcal{L}(X)$ to $\mathcal{L}(Y)$ that generalizes the (normalized) derivative. Then we explore some basic facts about rotational components. We conclude with a new characterization of the pre-compact translation surfaces.

3.1. **Affine maps.** Since $\mathbb{C} \cong \mathbb{R}^2$, every translation surface (X, ω) has a canonical *real-affine* translation structure.

Definition 3.1. Let (X, ω) and (Y, η) be translation surfaces. An open map $f : X \to Y$ is called affine if and only if $f : X' \to Y'$ is affine in real-affine charts. That is, in local translation coordinates, $f(x, y) = A \cdot (x, y) + (x_0, y_0)$ for some $A \in \mathrm{GL}(2, \mathbb{R})$.

The constant (x_0, y_0) depends on the local coordinates, but the differential $A \in \mathrm{GL}(2, \mathbb{R})$ does not, since all transition functions involved are translations. We denote by $\mathrm{Aff}(X, Y)$ the set of affine maps from X to Y, by $\mathrm{Aff}(X)$ the group of affine diffeomorphisms of X' and by $\mathrm{Aff}^+(X)$ the subgroup of $\mathrm{Aff}(X)$ formed by orientation preserving maps. Remark that every $f \in \mathrm{Aff}(X, Y)$ has a unique continuous extension $\hat{f} : \widehat{X} \to \widehat{Y}$. The action of $\mathrm{Aff}(X)$ extends naturally to $\widehat{X}$. Henceforth we denote by $\mathrm{Stab}(x)$ the stabilizer of $x \in \widehat{X}$ with respect to this action. Recall that the Veech group of (X, ω) is $\Gamma(X) := \{Df \in \mathrm{GL}(2, \mathbb{R}) \mid f \in \mathrm{Aff}(X)\}$. Every countable subgroup of $\mathrm{GL}_+(2, \mathbb{R})$ without elements of norm less than 1 can be realized the Veech group of a translation surface having only finite angle singularities, infinite genus and one end (see [PSV]).

The map f_*. Every $f \in \mathrm{Aff}(X, Y)$ sends flat points in X to flat points in Y. For every $\gamma \in \mathcal{L}^{\varepsilon}(X)$ let ε' be the total length of $f \circ \gamma$. Then there exists a unique unit speed geodesic $\eta \in \mathcal{L}^{\varepsilon'}(X)$ parametrizing the image of $f \circ \gamma$ and such that $\mathrm{bp}[\eta] = \lim_{t \to 0} f \circ \gamma(t)$. We define $f_* : \mathcal{L}(X) \to \mathcal{L}(Y)$ as $f_*[\gamma] := [\eta]$.

This definition does not depend on the representative γ. Remark that $(f \circ g)_* = f_* \circ g_*$ and $(\mathrm{id}_X)_* = \mathrm{id}_{\mathcal{L}(X)}$. The following theorem implies that $(\mathcal{L}(\,\cdot\,), (\,\cdot\,)_*)$ is a functor from the category of translation surfaces satisfying the main hypotheses (see §1) with affine maps to **Top**, the category of topological spaces.

Theorem 3.2. *If $f : X \to Y$ is affine, then $f_* : \mathcal{L}(X) \to \mathcal{L}(Y)$ is continuous.*

Proof. Define $f_\varepsilon : \mathcal{L}^{\varepsilon}(X) \to \mathcal{L}(Y)$ as $f_\varepsilon(\gamma) := f_*[\gamma]$. By the universal property of the direct limit, it is sufficient to prove that for every $0 < \varepsilon' < \varepsilon$ the map f_ε is continuous and $f_\varepsilon = \rho_\varepsilon^{\varepsilon'} \circ f_{\varepsilon'}$, where $\rho_\varepsilon^{\varepsilon'} : \mathcal{L}^{\varepsilon}(X) \to \mathcal{L}^{\varepsilon'}(X)$ denotes the restriction map. The equation $f_\varepsilon = \rho_\varepsilon^{\varepsilon'} \circ f_{\varepsilon'}$ is clear from the definition of f_*. Recall that, by Lemma **??** in the preceding section, the sets $U = \bigcap_{i=1}^{n} \tilde{B}(y_i, r_i)^{t_i}$ form a basis for the topology of $\mathcal{L}(Y)$. Let $\gamma \in f_\varepsilon^{-1}(U)$, $f_\varepsilon(\gamma) = [\eta]$ and $s_i \in \mathbb{R}$ be such that $f \circ \gamma(s_i) = \eta(t_i)$, $i = 1, \ldots, n$.

For every $\gamma \in \mathcal{L}^{\varepsilon}(X)$ denote by $|f \circ \gamma|$ the total length of the image of $f \circ \gamma$. Since f is affine (in particular quasiconformal), for every $r > 0$ there exists $\rho > 0$ such that, for every γ_1 in a d_ε-neighborhood of radius ρ around γ, one has that $|f \circ \gamma_1| = K |f \circ \gamma|$, with $K = K(\gamma_1)$ and $|K - 1| < r$. Furthermore, by the continuity of f, such a $\rho > 0$ can be chosen so that $d_Y(f \circ \gamma_1(s_i), f \circ \gamma(s_i)) < r$ for every

$i = 1, \dots, n$. Let $[\eta_1] := f_\varepsilon(\gamma_1)$. Remark that $f \circ \gamma_1(s_i) = \eta_1(t_i')$, where $t_i' = Kt_i$. Hence, if $r > 0$ is small enough, then η_1 is defined at $t = t_i$ and $\eta_1(t_i) \in B(y_i, r_i)$ for every $i = 1, \dots, n$. $\square$

Remark 3.3. Every map $f \in \mathrm{Aff}(X, Y)$ acts on the space of directions S^1 by its normalized differential. In proposition 2.12, we proved that there is a natural topological emmbeding $i : T_1(X') \hookrightarrow \mathcal{L}(X)$. Every class $[\gamma] \in i(T_1(X'))$ is completely determined by the pair $(\mathrm{bp}(\gamma), \mathrm{dir}(\gamma))$. By definition the class of $f_*[\gamma]$ is determined by $(\mathrm{bp}(f_*[\gamma]), \frac{Df}{|Df|}(\mathrm{dir}(\gamma))$. In other words, the map f_* is the continuous extension to $\mathcal{L}(X)$ of the normalized derivative of the affine map f. Remark that the preceding theorem does not follow from the classical extension theorems for continuous maps, since $\mathcal{L}(X)$ is in general not regular. On the other hand, remark that $\hat{f} \circ \mathrm{bp}[\gamma] = \mathrm{bp} \circ f_*[\gamma]$ for every $[\gamma] \in \mathcal{L}(X)$.

Corollary 3.4. *If $f : X \to Y$ is an affine homeomorphism, then $f_* : \mathcal{L}(X) \to \mathcal{L}(Y)$ is a homeomorphism.*

Remark 3.5. The converse of this statement does not hold by any means. Even in the case of surfaces of finite affine type, if X and Y are in the same connected component of a stratum, then $\mathcal{L}(X)$ is homeomorphic to $\mathcal{L}(Y)$, but this homeomorphism is only induced by an affine map $X \to Y$ if they lie in the same $\mathrm{SL}_2(\mathbb{R})$-orbit.

Corollary 3.6. *There is a canonical injection from* $\mathrm{Aff}(X)$ *into* $\mathrm{Homeo}(\mathcal{L}(X))$ *and from* $\mathrm{Stab}(x)$ *into* $\mathrm{Homeo}(\mathcal{L}(x))$ *for every* $x \in \widehat{X}$.

3.2. **Action of f_* on rotational components.** Recall from the preceding section that the restriction to f_* to $T^1(X')$ is given by the *normalized differential* $\frac{Df}{|Df|}$. In particular, the action of f_* on a rotational component of $\mathcal{L}(x)$, with $x \in X'$, is given by $\frac{Df}{|Df|}$ as well. This situation is not proper to flat points.

Lemma 3.7. *Let $x \in \widehat{X} \setminus X'$, $\overline{[\gamma]}$ be a rotational component in $\mathcal{L}(x)$ and $\overline{[\eta]}$ its image under $f_{*|} : \mathcal{L}(x) \to \mathcal{L}(y)$. Denote by $A_f := \frac{Df}{|Df|}$. Then the following diagram commutes:*

$$\begin{tikzcd} \overline{[\gamma]} \arrow[r, "f_{*|}"] \arrow[d, "\mathrm{dir}"] & \overline{[\eta]} \arrow[d, "\mathrm{dir}"] \\ S^1 \arrow[r, "A_f"] & S^1 \end{tikzcd} \tag{3.5}$$

Proof. Let $[\gamma] \in \overline{[\gamma]}$. If $\varepsilon > 0$ is small enough there exists a representantive $\gamma : (0, \varepsilon) \to X'$ and flat charts (U, φ) and (V, ψ) of the image of γ and $f \circ \gamma$ in X' and Y' respectively, for we are assuming that the set of singularities of any translation surface is discrete. Without loss of generality we can suppose that $0 = \lim_{t\to 0} \varphi(\gamma(t)) = \lim_{t \to 0} \psi(f(\gamma(t)))$. That is, up to composing with a translation, the action in local coordinates of f on γ is linear. Since the differential of f does not depend on local coordinates and we can rescale both vectors without changing the direction, we obtain the commutativity of (3.5). $\square$

Let $[\gamma_0] \in \overline{[\gamma]}$ and $f_{*|}[\gamma_0] = [\eta_0]$. Denote by $\alpha_0 = \mathrm{dir}[\gamma_0]$, $\beta_0 = \mathrm{dir}[\eta_0]$ and by $\exp : \mathbb{R} \to S^1$ the universal covering map. Choose $t_0 \in \exp^{-1}(\alpha_0)$ and $s_0 \in \exp^{-1}(\beta_0)$. There is a unique lift $\widetilde{A_f} : \mathbb{R} \to \mathbb{R}$ of $A_f : S^1 \to S^1$ sending t_0 to s_0. Moreover,

there is a unique translation embedding $i_0 : \overline{[\gamma]} \hookrightarrow \mathbb{R}$ such that $i_0([\gamma_0]) = t_0$ and making the following diagram commute:

$$\begin{tikzcd} \overline{[\gamma]} \arrow[r, hook, "i_0"] \arrow[d, "\mathrm{dir}"] & \mathbb{R} \arrow[d, "\exp"] \\ S^1 \arrow[r, "Id"] & S^1 \end{tikzcd} \tag{3.6}$$

The same is valid for a translation embedding $j_0 : \overline{[\eta]} \hookrightarrow \mathbb{R}$ satisfying $j_0([\eta_0]) = s_0$. Hence, if we think of $\mathbb{R}$ as local coordinates for the rotational components $\overline{[\gamma]}$ and $\overline{[\eta]}$, the action of f_* on a rotational component is described globally by the following equation:

$$f_*|_{\overline{[\gamma]}} = (j_0^{-1} \circ \widetilde{A_f} \circ i_0) \tag{3.7}$$

Definition 3.8. We call an area-preserving affine automorphism of a flat surface parabolic, elliptic, or hyperbolic according to whether the image of Df in $\mathrm{PSL}(2, \mathbb{R})$ is parabolic, elliptic or hyperbolic, respectively.

Definition 3.9. Let $\overline{[\gamma]}$ be a rotational component with non-empty boundary. We call $\alpha \in S^1$ a limit direction of the rotational component if it is the limit of the map $\mathrm{dir}_| : \overline{[\gamma]} \to S^1$ as one approaches the boundary point of $\overline{[\gamma]}$.

Proposition 3.10. *Suppose that the singular locus* $\mathrm{Sing}(X)$ *of* $\widehat{X}$ *is finite and that there exists a rotational component* $\overline{[\gamma]}$ *of finite length* λ.

1. *If* $f \in \mathrm{Aff}(X)$ *is parabolic and* $\lambda \not\equiv 0 \pmod{\pi}$, *or*
2. *If* $f \in \mathrm{Aff}(X)$ *is hyperbolic and the limit directions of* $\overline{[\gamma]}$ *are not invariant under* A_f, *or*
3. *If* $f \in \mathrm{Aff}(X)$ *is elliptic but its image in* $\mathrm{PSL}(2, \mathbb{R})$ *is not conjugated to a torsion element,*

then there exists $x_0 \in \mathrm{Sing}(X)$ *such that* $\mathcal{L}(x_0)$ *has an infinite number of finite length rotational components.*

Proof. We proceed by contradiction. Without loss of generality we can suppose that there is a point $x \in \mathrm{Sing}(X)$ such that the rotational component $\overline{[\gamma]} \in \mathcal{L}(x)$ is fixed by $f_{*|}$. If f is parabolic, $\widetilde{A_f} : \mathbb{R} \to \mathbb{R}$ is a map whose fixed points form a lattice of the form $\pi\mathbb{Z} + t$, for some $t \in \mathbb{R}$. In particular, it does not preserve the length of any subinterval I whose endpoints are not in the lattice, which is always the case if $\lambda \not\equiv 0 \bmod(\pi)$. If f is hyperbolic, the fixed points of $\widetilde{A_f}$ form two lattices $\pi\mathbb{Z} + t$, $\pi\mathbb{Z} + s$ for some real numbers $s \neq t$. In particular it does not preserve the length of any subinterval I whose endpoints are not in the union of this two latices. Such is the case if the limit directions of $\overline{[\gamma]}$ are not invariant under A_f. If f is elliptic but its image in $\mathrm{PSL}(2, \mathbb{R})$ is not conjugated to a torsion element, then no power of A_f fixes a direction in S^1. In particular, it cannot fix the limiting directions of $\overline{[\gamma]}$. □

3.3. **Rotational components.** In this subsection we state and prove some basic facts about rotational components. Then we provide a method to detect when a linear approach is in the boundary of the rotational component it defines and we discuss transverse measures on subsets of $\mathcal{L}(X)$. Finally, we present a characterization for pre-compact translation surfaces in the language developed in this article.

Through the examples in §1.1, we showed the existence of rotational components isometric to $\mathbb{R}$, open intervals, and points. In fact, it is not difficult to combine ideas from these examples to realize any connected subset of the real line as a rotational component. We can detect when a linear approach is in the boundary of the rotational component it defines. For this we introduce the continuous function $r : X' \to \mathbb{R}^+ \cup \infty$ defined by

$$r(x) := \sup\{r > 0 \mid B(x,r) \subset X'\} = \mathrm{dist}(x, \mathrm{Sing}(X)). \tag{3.8}$$

That is, $r(x)$ is the largest radius of a disk immersed in X' and centered at x.

Lemma 3.11. *For every $x \in \widehat{X}$, there exists a rotational component in $\mathcal{L}(x)$ without empty interior.*

Proof. If x is a flat point, then $\mathcal{L}(x) \cong S^1$ and the result is clear. So suppose $x \in \widehat{X}$ is a singular point. By our main hypothesis, there exists $r > 0$ such that $B(x,r) \cap \mathrm{Sing}(X) = \{x\}$. Let $y \in B(x,r)$ be a flat point. Then $r(x) < r$ and $x \in \partial B(y, r(x))$. Let $\gamma : (0, r(x)) \to X'$ be the linear approach contained in the segment joining y to $\lim_{t\to 0} \gamma(t) = x$. Then $\overline{[\gamma]}$ contains an open interval centered at $[\gamma]$ of length at least π. □

Lemma 3.12. *A class $[\gamma] \in \mathcal{L}(X)$ is in the boundary of a rotational component if and only if it contains a representative γ such that such that for every $M > 0$ there exists $t \in (0, \varepsilon)$ for which $r(\gamma(t)) < Mt$.*

Proof. We address first sufficiency. Let $\gamma : (0, \varepsilon) \to X'$ be a representative for which there is $M > 0$ such that for every $t \in (0, \varepsilon)$ we have $(r \circ \gamma)(t) \geq Mt$. Then, there exists an open subset D of $\cup_{t\in(0,\varepsilon/2)} B(\gamma(t), r(\gamma(t)))$ such that: (i) there is a flat chart defined on D onto the xy-plane for which $\gamma(t)$, $t \in (0, \varepsilon/2)$ is the segment $(0, \varepsilon)$ and (ii) in this coordinates the segments $\{(s, M's) \mid s \in (0, \varepsilon/2),\ 0 < M' < M/2\}$ define an open $U \subset D$ in $\overline{[\gamma]}$ containing $[\gamma]$.

For necessity remark that if $[\gamma]$ is an interior point of $\overline{[\gamma]}$, then there is an angular sector (I, c, i_c), with I an open interval, such that $i_c(U(I,c))$ contains a representative $\gamma : (0, \varepsilon) \to X'$ of $[\gamma]$. Without loss of generality we suppose that such representative corresponds to the middle point of I. Given that the image of i_c lies within X', we can think of $i_c(U(I,c))$ as the angular sector in the xy-plane defined by:

$$\{(x,y) \mid 0 < x < \varepsilon,\ |y| \leq Mx\}$$

for some fixed $M > 0$. Hence $(r \circ \gamma)(t) \geq Mt$ for all $t \in (0, \varepsilon)$. □

Proposition 3.13. *Each $\mathcal{L}(x)$ is the closure of the interior of its rotational components.*

Proof. Let $[\gamma] \in \mathcal{L}(x)$. If $[\gamma]$ itself is contained in the interior of a rotational component or if it is a boundary point for some rotational component with non-empty interior, then we are done. Suppose, then, that it is not, and choose a representative path $\gamma : (0, \varepsilon) \to X$. Consider a segment $\sigma(t)$ of variable length traveling along γ, for which $\gamma(t)$ is its perpendicular bisector at each time t. By Lemma 3.12, $(r \circ \gamma)(t) \to 0$ as $t \to 0$. We may assume that the length of $\sigma(t)$ is a monotonic, piecewise constant function along $(0, \varepsilon)$ (necessarily having countably infinitely many discontinuities) that tends to 0 at $t \to 0$. Let $s(t)$ be the maximal convex function bounded above by the length function of $\sigma(t)$; this is a continuous

piecewise affine function with countably many points of non-differentiability, corresponding to a subset of times when $\sigma(t)$ reaches a singularity. Then for every $t \in (0, \varepsilon)$, there is at least one trajectory γ_t having the same direction as γ and basepoint at the point of $\sigma(t)$ at distance $s(t)$ from $\gamma(t)$ along $\sigma(t)$. Each of these γ_t lies in the interior of a rotational component (by convexity of $s(t)$). Moreover, the $[\gamma_t]$ converge to $[\gamma]$. By taking times $t_n \to 0$ where $s(t)$ is non-differentiable, we obtain a sequence $[\gamma_{t_n}] \to [\gamma]$ of linear approaches having basepoints in $\operatorname{sing}(X)$. Because x is isolated from the rest of $\operatorname{Sing}(X)$, the result is proved. $\square$

Remark 3.14. Each rotational component is naturally immersed in $\mathcal{L}(X)$. However, its topology as translation 1-manifold can differ from its topology as subspace of $\mathcal{L}(X)$. Consider the geometric construction (§1.1, example 1.12) performed on the real plane. The resulting translation surface presents just one wild singularity x and $\mathcal{L}(x)$ is composed by two rotational components each of which is, as translation 1-manifold, isometric to $(0, \infty)$. Let $\gamma_1(t) = (1, t)$ and $x = (1, \frac{1}{2})$. It is not difficult to see that for each $0 < r < \frac{1}{4}$ the open set $\widetilde{B}(x, r)^{\frac{1}{2}}$ contains $\overline{[\gamma_1]}$ and $\widetilde{B}(x, r)^{\frac{1}{2}} \cap \overline{[\gamma_1]}$ is formed by an infinite family of disjoint open intervals $\{(a_n, b_n)\} \subset (0.\infty)$. Given that $\overline{[\gamma_1]}$ as a 1-manifold is *locally connected*, the preceding discussion implies that the topology of $\overline{[\gamma_1]}$ as subspace of $\mathcal{L}(X)$ is not equivalent to its topology as a 1-manifold.

3.4. **Transverse measures.** Along with the angular metric on each rotational component of $\mathcal{L}(X)$ and the length metric on subsets of $\mathcal{L}(X)$ lying along the same geodesic trajectory, there is an obvious measure to consider on subsets of $\mathcal{L}(X)$ having the same direction. For each $\theta \in S^1$, let $\mathcal{L}_\theta(X) = \operatorname{dir}^{-1}(\theta)$, and for each $x \in \widehat{X}$, let $\mathcal{L}_\theta(x) = \mathcal{L}_\theta(X) \cap \mathcal{L}(x)$. We use $\mathcal{F}_\theta$ to denote the foliation of X in the direction θ. Elements of $\mathcal{L}_\theta(X)$ may now be thought of as germs of (oriented) leaves of $\mathcal{F}_\theta$.

Recall that $\mathcal{F}_\theta$ is obtained by integrating the kernel field of the one-form $v \mapsto v \cdot v_{\theta^\perp}$, where $v_{\theta^\perp}$ is the vector field of unit-length vectors whose direction is rotated $\pi/2$ counterclockwise from the direction θ; $\mathcal{F}_\theta$ carries the transverse measure ν_θ, which is the absolute value of this one-form. If $f : X \to \mathbb{R}_{\geq 0}$ is any non-negative, locally bounded, Borel measurable function, then $f\nu_\theta$ is a Borel measure that can be integrated, at least, over rectifiable curves in X.

Let $B \subset \mathcal{L}_\theta(X)$ be a Borel subset. A *representative* of B will be a continuous choice L_B of representatives for $[\gamma] \in B$, i.e., a continuous section $B \to \tilde{\mathcal{L}}(X)$, so that $L_B([\gamma]) \in [\gamma]$ varies continuously in length with respect to the topology on $\tilde{\mathcal{L}}(X)$. The *length* of a representative L_B (which may be infinite) is

$$\ell(L_B) = \sup_{[\gamma] \in B} \{\operatorname{length}(L_B([\gamma]))\}.$$

A piecewise C^1 curve $\tau : I \to X$ (I may be open or closed, bounded or unbounded) is said to be *transverse* to a representative L_B if it is transverse to each element of L_B; a collection of at most countably many piecewise C^1 curves $\{\tau_i\}$ is *full* with respect to L_B if each τ_i is transverse to L_B and every element of L_B intersects some τ_i. Each representative L_B of B induces a characteristic function $\chi_B : X \to \{0, 1\}$ whose support is the union of the images of elements of L_B. (Note that χ_B is not

canonical; it depends on a choice of L_B.) Now we define the measure μ_θ of B by

$$\mu_\theta(B) = \limsup_{\ell(L_B)\to 0} \left[\inf \left\{ \sum_i \int_{\tau_i} \chi_B \, \nu_\theta \;\middle|\; \chi_B \text{ induced by } L_B,\ \{\tau_i\} \text{ full with respect to } L_B \right\} \right].$$

It is fairly obvious that μ_θ is a Borel measure on $\mathcal{L}_\theta(X)$. Likewise, it restricts to a Borel measure on $\mathcal{L}_\theta(x)$ for any $x \in \widehat{X}$. The measures μ_θ and ν_θ are compatible in the following sense.

Proposition 3.15. *Let σ be a topological segment in X, transverse to $\mathcal{F}_\theta$, and let $\tilde{\sigma} = (\sigma, \theta)$ be the corresponding subset of $X \times S^1 \subset \mathcal{L}(X)$. Then $\tilde{\sigma}$ is a Borel subset of $\mathcal{L}_\theta(X)$, and $\nu_\theta(\sigma) = \mu_\theta(\tilde{\sigma})$.*

Proof. Clear. □

Remark 3.16. The three types of measures on subsets of $\mathcal{L}(X)$—rotational, geodesic, and transverse—are closely analogous to the three types of closed one-parameter subgroups of $\mathrm{SL}(2,\mathbb{R})$. Indeed, rotational components are the orbits of a canonical "partial action" on $\mathcal{L}(X)$ by $\widetilde{\mathrm{SO}}(2)$, the universal cover of $\mathrm{SO}(2)$. The motions along and transversely to geodesic trajectories are akin to the geodesic and horocyclic flow on a hyperbolic surface.

3.5. **Finite type surfaces.** To conclude this section, we provide a new characterization of the "classical" translation surfaces, which are included in the next definition.

Definition 3.17. A translation surface has *finite affine type* if it has finite area and the underlying Riemann surface has finite analytic type (that is, it is obtained from a compact Riemann surface by finitely many punctures).

These are often called "pre-compact" translation surfaces in the literature (see [GJ00]); however, as the examples in §1.1 show, a translation surface of infinite genus may have a metric completion which is compact. The condition of finite area is necessary to rule out abelian differentials which are holomorphic on the surface but have poles at the punctures.

Lemma 3.18. *Let $x \in \widehat{X}$. Suppose ℓ has a positive lower bound on $\mathcal{L}(x)$. Then x is either a cone point or an infinite-angle singularity. In particular, if $\mathcal{L}(x)$ is compact, then x is a cone point.*

Proof. If ℓ has a positive lower bound on $\mathcal{L}(x)$, then the direction map $\mathcal{L}(x) \to S^1$ is a covering map, from which the result follows. □

Proposition 3.19. *X has finite affine type if and only if $\widehat{X}$ is compact and $\mathcal{L}(x)$ is compact for every $x \in \widehat{X}$.*

Proof. Suppose that X has finite affine type. Then X may be made into a compact Riemann surface $\tilde{X}$ by adding finitely many points. The translation structure on X is given by an abelian differential, which, because it has finite area, extends to an abelian differential on $\tilde{X}$. In this way, $\widehat{X}$ is canonically homeomorphic to $\tilde{X}$, hence compact, and every point of $\widehat{X}$ is either a regular point or a cone point, which implies that $\mathcal{L}(x) \cong S^1$ for every x.

Conversely, assume that $\widehat{X}$ is compact and $\mathcal{L}(x) \cong S^1$ for every $x \in \widehat{X}$. Because ℓ is lower semicontinuous, it has a positive lower bound on each $\mathcal{L}(x)$. This implies, from the preceding lemma, that the singularities of $\widehat{X}$ are all cone points or marked points. Therefore the conformal structure of X extends to $\widehat{X}$, and the cone points of X are all finite-order zeroes of an abelian differential on $\widehat{X}$, which means X has finite affine type. □

4. Isometries between neighborhoods of singularities

Let X and Y be translation surfaces. In this section, we will describe a set of necessary and sufficient conditions for $x \in \widehat{X}$ and $y \in \widehat{Y}$ to have isometric (or more precisely, translation equivalent) neighborhoods. Given $\varepsilon > 0$, let $N_\varepsilon(x)$ and $N_\varepsilon(y)$ denote the ε-neighborhoods of x and y, respectively, and set $N'_\varepsilon(x) = N_\varepsilon(x) \setminus \{x\}$ and $N'_\varepsilon(y) = N_\varepsilon(y) \setminus \{y\}$. We assume throughout this section that any choice of ε is made so that $N'_\varepsilon(x)$ and $N'_\varepsilon(y)$ contain no singularities; this is possible by our standing assumption that the singular sets of X and Y are discrete.

Recall that we have defined the direction function dir from both $\mathcal{L}(x)$ and $\mathcal{L}(y)$ to S^1, the maximal length function ℓ from $\mathcal{L}(x)$ and $\mathcal{L}(y)$ to $(0, \infty]$, and the transverse measures μ_θ on $\mathcal{L}_\theta(x)$ and $\mathcal{L}_\theta(y)$ for all $\theta \in S^1$. For our present purposes, we must localize the notion of maximal length. Given $\varepsilon > 0$, let

$$\ell_\varepsilon[\gamma] = \min\{\varepsilon, \ell[\gamma]\}.$$

Also let σ_ε be the involution defined on $\ell^{-1}((0,\varepsilon))$ in $\mathcal{L}(x)$ or $\mathcal{L}(y)$ by

$$\sigma_\varepsilon([\gamma(t)]) = [\gamma(\ell(\gamma) - t)].$$

This is the "pairing" function on short saddle connections, since each saddle connection on from x to itself, for instance, defines two elements of $\mathcal{L}(x)$.

Theorem 4.1. *Let $x \in \widehat{X}$ and $y \in \widehat{Y}$. Then the following are equivalent:*

(1) *There exist $\varepsilon > 0$ and a homeomorphism $F : \mathcal{L}(x) \to \mathcal{L}(y)$ such that*

$$\ell_\varepsilon \circ F = \ell_\varepsilon, \qquad \sigma_\varepsilon \circ F = F \circ \sigma_\varepsilon,$$
$$\mathrm{dir} \circ F = \mathrm{dir}, \qquad \textit{and} \qquad \forall\ \theta \in S^1,\ F^*\mu_\theta = \mu_\theta.$$

(2) *There exist $\varepsilon' > 0$ and a translation equivalence $N'_{\varepsilon'}(x) \to N'_{\varepsilon'}(y)$.*

Remark 4.2. When x and y are flat points or cone points, this theorem reduces to the statement that x and y have isometric neighborhoods if and only if they have the same total angle.

The implication "(2) $\implies$ (1)" in Theorem 4.1 is obvious by taking $\varepsilon = \varepsilon'$, so assume (1) holds; we will show (2) holds with $\varepsilon' = \varepsilon/2$. The proof is by construction. Given $z \in N'_{\varepsilon/2}(x)$, set $\delta_z = d_X(x, z)$, and let γ_z be a shortest trajectory from x to z, meaning $\gamma_z(\delta_z) = z$. Note that this implies $\ell[\gamma_z] > 2\delta_z$. When $F([\gamma]) = [\eta]$, we will write $F(\gamma)(t)$ in place of $\eta(t)$ to avoid introducing new symbols. In the proof of the following lemma, we also use $[\gamma] + \theta$, for $\theta \in \mathbb{R}$, to mean the linear approach $[\eta]$ in the rotational component $\overline{[\gamma]}$ such that, with respect to the translation structure on $\overline{[\gamma]}$, $[\gamma]$ and $[\eta]$ differ by θ, when such an $[\eta]$ exists.

Lemma 4.3. *With the above assumptions, $F(\gamma_z)$ is a shortest path from y to $F(\gamma_z)(\delta_z)$.*

Proof. The assumption that γ_z is a shortest path from x to z implies that it lies in a rotational component of $\mathcal{L}(x)$ having length at least π (see proof of Lemma 3.11), and $\ell([\gamma_z] + \theta) \geq 2\delta_z \cos\theta$ for $|\theta| < \pi/2$. Because $\varepsilon > 2\delta_z$ and F preserves ℓ_ε, we have $\ell(F([\gamma_z]) + \theta) \geq 2\delta_z \cos\theta$ for $|\theta| < \pi/2$, which shows that the maximal immersed disk centered at $F(\gamma_z)(\delta_z)$ also has radius δ_z. □

For each $z \in N'_{\varepsilon/2}(x)$, choose a shortest path γ_z from x to z, and set $w = F(\gamma_z)(\delta_z)$.

Lemma 4.4. *The point $w \in N'_{\varepsilon/2}(y)$ is independent of the choice of γ_z.*

Proof. Let $B(z, \delta_z)$ be the δ_z-neighborhood of z. This neighborhood is the image of an immersed Euclidean disk $\tilde{B}(z, \delta_z)$. Let γ_1 and γ_2 be two choices for γ_z. Then the segments $\gamma_i((0, \delta_z))$ are radii of $B(z, \delta_z)$; let η be the saddle connection between the corresponding points of $\partial\tilde{B}(z, \delta_z)$. Set $w_i = F(\gamma_i)(\delta_z)$; we want to show that $w_1 = w_2$. For this it suffices to show that the (immersed) triangle formed by γ_1, γ_2, and η is sent to a congruent triangle. This follows from the assumptions $\ell_\varepsilon \circ F = \ell_\varepsilon$, $\sigma_\varepsilon \circ F = F \circ \sigma_\varepsilon$, and $\mathrm{dir} \circ F = \mathrm{dir}$, together with the ASA congruence theorem. □

Lemma 4.4 implies that we can define $f : N'_{\varepsilon/2}(x) \to N'_{\varepsilon/2}(y)$ unambiguously by

$$f(z) = F(\gamma_z)(\delta_z).$$

Lemma 4.5. *The map f is a bijection.*

Proof. First observe that, by Lemma 4.3, the construction of f can be applied in the reverse direction using F^{-1} to obtain an inverse map f^{-1}. Thus every point of $N'_{\varepsilon/2}(y)$ is covered via f by a point of $N'_{\varepsilon/2}(x)$, which shows that f is surjective. To see that it is injective, note that this is just Lemma 4.4 applied in the reverse direction, *i.e.*, it is the observation that f^{-1} is well-defined. □

Lemma 4.6. *The map f is a local isometry.*

Proof. Let $z \in N'_{\varepsilon/2}(x)$. We want to show that some embedded disk centered at z is carried isometrically into $N'(y)$. The idea of our proof is to consider "polar coordinates" at z and to show that these are preserved by f. Let D_z be the largest open, embedded disk centered at z and contained in $N'_{\varepsilon/2}(x)$; its radius is therefore $\min\{\varepsilon/2 - \delta_z, \mathrm{inj\,rad}(z)\}$, where $\mathrm{inj\,rad}(z)$ is the injectivity radius of X at z (this is at most δ_z).

Let γ_z be a shortest path from x to z. Given $z' \in D_z$, let $\gamma_{z'}$ be a shortest path from x to z' and let $[z, z']$ denote the (unique) shortest segment from z to z'. If γ_z and $\gamma_{z'}$ lie in the same rotational component and are rotations of each other by an angle $< \pi/2$, then the segments γ_z, $[z, z']$, and $\gamma_{z'}$ form the sides of a Euclidean triangle, which is sent to a congruent triangle by construction (and an application of the SAS congruence theorem). Otherwise, we construct a path that is a union of saddle connections and is carried isometrically to $N'_{\varepsilon/2}(y)$.

Let $B(z, \delta_z)$ and $B(z')$ be the open δ_z and $\delta_{z'}$ neighborhoods of z and z', respectively; use the developing map of X to lift these to overlapping disks $\widetilde{B}(z)$ and $\widetilde{B}(z')$ in the plane. By assumption, $\partial\widetilde{B}(z)$ and $\partial\widetilde{B}(z')$ each have points that map to x; call these x_1 and x_2. If there is a path in $\widetilde{B}(z) \cup \widetilde{B}(z')$ between these points, then its length is less than ε, and we are done: a quadrilateral is determined by the lengths of three of its sides and the angles between them, and these data are

preserved by f. If no such path exists, then the segment in the plane from x_1 to x_2 passes outside of $\widetilde{B}(z) \cup \widetilde{B}(z')$. Let η be the shortest path in X homotopic to the union of γ_z, $[z, z']$, and $\gamma_{z'}$, relative to its endpoints. Then the lift of η to the plane by the developing map is a piecewise linear curve $\tilde{\eta}$, with possibly infinitely many points of non-differentiability, occurring at other points that project to x along η; at each such point, η turns consistently to either the right or the left. Each of these angles is preserved by f. Thus we only need to confirm that the lengths of the straight segments of $\tilde{\eta}$ are preserved. Each such segment σ is a union of saddle connections (with length $< \varepsilon$) and points that project to x. The lengths, direction, and pairing of the saddle connections along σ are preserved. Length is also preserved along all critical trajectories emanating from x in the direction θ perpendicular to σ. This set of trajectories forms a closed subset of $\mathcal{L}_\theta(x)$, so its transverse measure is preserved by f. Thus the total length of σ is preserved. We conclude that η is sent by f to an isometric path in $N_{\varepsilon/2}(y)$.

Thus we see that $d_X(z, z') = d_Y(f(z), f(z'))$, and the angle between γ_z and $[z, z']$ equals the angle between $F(\gamma_z)$ and $[f(z), f(z')]$. This proves the result. $\square$

Proof of Theorem 4.1. A bijection between Riemannian manifolds that is a local isometry is also an isometry, and so the result follows immediately from the preceding lemmas. $\square$

5. Final remarks

In this section we compare the space of directions and the Alexadrov cone of $\widehat{X}$ at a singular point x to $\mathcal{L}(x)$. Both are metric spaces used to extract information from a neighborhood of a point in a metric space. For a more detailed exposition on these objects we refer the reader to [BBI01].

Recall that the *space of directions* at a point $x \in \widehat{X}$ is the metric space consisting of curves emanating from x for which a comparison angle exists. The corresponding metric is the *upper angle* metric $\measuredangle_U(\cdot, \cdot)$.

Let $\widehat{X}$ be the metric completion of a finite cyclic covering of $\mathbb{C}^*$ and $x_0 = \mathrm{Sing}(X)$. We index the sheets of this covering by $\mathbb{Z}/n\mathbb{Z}$. Let γ_1 and γ_2 be two linear approaches to x_0 whose images do not lie in the same sheet of the covering. That is, their distance in the rotational component forming $\mathcal{L}(x_0)$ is greater than 2π. A straightforward calculation shows that $\measuredangle_U(\gamma_1, \gamma_2) = \pi$. Hence with the space of directions we obtain less information about the set of geodesics emanating from x_0 than with $\mathcal{L}(x_0)$.

Let Γ_x denote the set of geodesics emerging from a singularity $x \in \widehat{X}$. Define on $\Gamma_x \times [0, \infty)$ the pseudo-metric:

$$d\big((\gamma_1, s_1), (\gamma_2, s_2)\big) := \limsup_{t \to 0^+} \frac{d_X(\gamma_1(ts_1), \gamma_2(ts_2))}{t} \tag{5.9}$$

Let C_x denote the metric space obtained after taking the quotient by (5.9) and $\widehat{C_x}$ the corresponding metric completion. This metric space is called the *Alexandrov cone* at x. Suppose that x is an infinite angle singularity. It is not hard to find a pair of linear approaches $[\gamma_1] \neq [\gamma_2]$ to x and a pair of positive real numbers s_1, s_2 such that the distance (5.9) between the points (γ_1, s_1) and (γ_2, s_2) is arbitrarily small and at the same time the distance between $[\gamma_1]$ and $[\gamma_2]$ in the corresponding double spire is arbitrarily large. In other words, from the metric point of view

the Alexandrov cone cannot tell apart linear approaches in the same rotational component that are far away from each other.

More seriously, because the space of directions and the Alexandrov cone are metric spaces, they lose certain convergence information contained in $\mathcal{L}(x)$; see remark 2.10. However, the Alexandrov cone can be completely recovered from $\mathcal{L}(x)$ by separating into rotational components.

Remark 5.1. For "good" metric spaces, the Gromov–Hausdorff tangent cone at a point x_0 is nothing but the metric cone over the space of directions at x_0, where as for "bad" (*e.g.*, non locally compact, non boundedly compact) this cone might not be well defined (see [BBI01, §8.2] for more details). Hence, with the Gromov–Hausdorff cone at a singularity we obtain (if any) less information about the set of geodesics emanating from x_0 than with $\mathcal{L}(x_0)$.

Most of our constructions only rely on the fact that a translation surface is a Riemannian manifold; indeed, in many cases only an affine connection is required. It would be interesting to know if these constructions have applications in other areas.

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